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On A-Spaces and Their Relation to the Hobby–Rice Theorem

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If U is an n-dimensional subspace of $C[a, b]$ and w is a positive, continuous function on $[a, b]$, the Hobby–Rice theorem asserts that there exist points $a = x_0 < x_1 < \cdots < x_m < x_{m+1} = b$ such that $m \leqslant n$ and $\sum_{j=1}^{m+1} (-1)^j \int_{x_{j-1}}^{x_j} uw \, d\mu = 0$ for all $u \in U$. It is shown that for the canonical set $\{x_j\}_{j=1}^{m}$ to contain n points and be unique for all admissible w, it is necessary and sufficient that U satisfy the WT-property and the splitting property (i.e., if $u \in U$ and $u \equiv 0$ on $[c, d]$ where $a < c < d < b$, then $u\chi_{[a,d]} \in U$). A new proof is given for the previously known result that for the canonical set to contain n points, be unique, and have full rank relative to U for all admissible w, it is necessary and sufficient that U be an A-space. For a WT-space U with the splitting property, the canonical sets for WT-extensions of U are shown to interlace with the canonical set for U, and a formula for the rank of canonical sets for U is given. In addition, it is shown that every A-space on an interval has an A-space extension.

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1. Introduction

The following theorem of Hobby and Rice [1, 10] plays a fundamental role in the theory of L^1-approximation. Let U be an n-dimensional subspace of $L^1([a,b],\nu)$ where ν is a positive, finite, nonatomic Borel measure. Then there exist points $a=x_0<x_1<\cdots<x_m<x_{m+1}=b$ such that $m\leqslant n$ and

$$\sum_{j=1}^{m+1}(-1)^j\int_{x_{j-1}}^{x_j} u\,d\nu=0 \tag{1.1}$$

for all $u\in U$. In this paper, we shall be concerned with subspaces U of $C[a,b]$ and measures ν of the form $d\nu=w\,d\mu$ where μ denotes Lebesgue measure and w is in the class C^+ of positive continuous functions on $[a,b]$. In this case, we call $\{x_j\}_{j=1}^m$ a w-canonical set for U.

For $w\in C^+$, let $C_w[a,b]$ denote the space of continuous real-valued functions on $[a,b]$ endowed with the w-weighted L^1-norm: $\|f\|_w=\int_a^b|f|\,w\,d\mu$. One of the main consequences of the Hobby–Rice theorem arises in the possibility of finding best $\|\cdot\|_w$-approximations to certain functions in $C_w[a,b]$ from U by interpolating on a w-canonical set (see [12, Appendix B]). In this vein, three questions naturally arise regarding the nature of the w-canonical sets. In the first place, when do the w-canonical sets consist of n points? Micchelli [9] showed that if U is an n-dimensional weak Chebyshev (WT-) space (that is, each element of U has at most $n-1$ sign changes in (a,b)), then all w-canonical sets for U contain n points ($w\in C^+$). Later, Kroó [4] established a converse of Micchelli's result. Specifically, if U is not a WT-space, then for some $w\in C^+$ there is a w-canonical set for U containing less than n-points. Thus only the WT-spaces have "full" w-canonical sets for all $w\in C^+$.

Let U be an n-dimensional WT-space, $w\in C^+$, and $\{x_j\}_{j=1}^n$ be a w-canonical set for U. If f is in the convexity cone $C(U)$ of U (that is, $U+sp\{f\}$ is a WT-space), then every best $\|\cdot\|_w$-approximation to f from U interpolates f at the points x_j ($1\leqslant j\leqslant n$; see [12, Appendix B]). We shall call the dimension of $U|_{\{x_j\}_{j=1}^n}$ the *rank* of the w-canonical set $\{x_j\}_{j=1}^n$. The other two questions were raised by Pinkus [12, p. 211]. When is the w-canonical set unique and when does it have rank n? The second question addresses the ability to interpolate on w-canonical sets and thus to find best $\|\cdot\|_w$-approximations to functions in $C(U)$ by interpolation. The question of uniqueness addresses the possibility of finding w-canonical sets via iterative or perturbative methods. To this end, Sommer [16] showed that if a WT-space U is a uniqueness space in $C_w[a,b]$ ($w\in C^+$; that is, every $f\in C[a,b]$ has a unique best $\|\cdot\|_w$-approximation from U), then the w-canonical set for U is unique and has full rank. Also, Micchelli [9] gave a sufficient condition based on $C(U)$ for the uniqueness and the full rank of the w-canonical set. Recently, Kroó [6] obtained a necessary and sufficient condition for a WT-space to have locally unique w-canonical sets for all $w \in C^+$ (see Proposition 4.1) and a necessary and sufficient condition for the w-canonical set for a WT-space to be unique and of full rank for all $w \in C^+$. More recently, the authors [7] proved that Kroó's latter condition is equivalent to the A-property. The remaining open problem is that of characterizing those subspaces whose w-canonical sets are globally unique for all $w \in C^+$.

The first major result of this paper completely characterizes the n-dimensional spaces that have unique w-canonical sets consisting of n points for all $w \in C^+$. Our characterization is based on the "ingredients" of the A-property which we now define. We say that a finite dimensional subspace U of $C[a, b]$ satisfies the *A-property* or is an *A-space* if for all $u \in U \setminus \{0\}$ and continuous $\sigma\colon [a, b] \setminus Z(u) \to \{-1, 1\}$ there exists $v \in U \setminus \{0\}$ such that $v = 0$ μ-a.e. on $Z(u)$ and $\sigma v \geqslant 0$ on $[a, b] \setminus Z(u)$. Throughout this paper, $Z(u) = \{x \in [a, b] : u(x) = 0\}$ and $Z(U) = \{x \in [a, b] : u(x) = 0$ for all $u \in U\}$. The A-property became a focal point in the study of uniqueness of best L^1-approximation. Specifically, a finite dimensional subspace U of $C[a, b]$ is a uniqueness space in $C_w[a, b]$ for all $w \in C^+$ if and only if U satisfies the A-property (see [3, 5, 11, 13]). In [11, 12], Pinkus gave a "spline-like" structural characterization of the A-spaces, and, quite recently, Li [8] obtained a considerable simplification of Pinkus' result which we now describe. We say that U satisfies the *splitting property* provided that if $u \in U$ and $u \equiv 0$ on $[c, d]$ where $a < c < d < b$, then $u\chi_{[a,c]}, u\chi_{[d,b]} \in U$ where χ_J denotes the characteristic function of $J \subseteq [a, b]$. We say that U satisfies the *decomposition property* if $z \in Z(U) \cap (a, b)$ implies that $U = U_{[a,z]} \oplus U_{[z,b]}$ where $U_J = \{u \in U : u \equiv 0$ on $[a, b] \setminus J\}$ for $J \subseteq [a, b]$. The Pinkus–Li characterization is as follows. A finite dimensional subspace U of $C[a, b]$ is an A-space if and only if it satisfies the WT-, splitting, and decomposition properties. Our first result is as follows.

THEOREM 1.1. *Let U be an n-dimensional subspace of $C[a, b]$. Then*

(a) *the w-canonical sets for U contain n points for all $w \in C^+$ if and only if U satisfies the WT-property,*

(b) *the w-canonical set for U contains n points and is unique for all $w \in C^+$ if and only if U satisfies the WT- and splitting properties.*

(c) *the w-canonical set for U contains n points, is unique, and has rank n for all $w \in C^+$ if and only if U satisfies the WT-, splitting, and decomposition properties (that is, U is an A-space).*

Thus we completely answer Pinkus' queries for varying weight functions. Actually, Theoem 1.1 is somewhat overstated as (b) is new whereas (a) and

(c) were previously known (see [4, 6, 7, 9]). New and simple proofs of (a) and (c) will be given in this paper.

The spaces in (b), the WT-spaces satisfying the splitting property, include the A-spaces and those spaces of the form $U = \{vw : v \in V\}$ where V is an A-space and w is a nonnegative continuous function on $C[a, b]$ where $Z(w)$ is nowhere dense. However, there are WT-spaces with the splitting property that are not continuously weighted A-space. The study of WT-spaces with the splitting property may lend insight into $\|\cdot\|_w$-approximation from A-spaces where w is allowed to vanish only on nowhere dense sets. As such, we establish two properties of these spaces which are intimately involved in the proof of Theorem 1.1. The first is an interlacing result for the w-canonical set for a WT-space U with the splitting property and the w-canonical sets for any WT-extension of U.

THEOREM 1.2. *Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property. Let $w \in C^+$, $\{x_j\}_{j=1}^n$ be the w-canonical set for U, and $x_0 = a$ and $x_{n+1} = b$. If V is an $(n+1)$-dimensional WT-space that contains U and $\{y_j\}_{j=1}^{n+1}$ is a w-canonical set for V, then $x_{j-1} \leqslant y_j \leqslant x_j$ $(1 \leqslant j \leqslant n+1)$.*

Theorem 1.2 is sharp in that one cannot obtain strict interlacing and the absence of the splitting property does not yield the interlacing result in the given form. In Section 2, we shall prove Theorem 1.2 and sufficiency in Theorem 1.1b and demonstrate the stated sharpness of Theorem 1.1.

Our next result provides a formula for the rank of a w-canonical set for a WT-space with the splitting property.

THEOREM 1.3. *Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property, $w \in C^+$, and $\{x_j\}_{j=1}^n$ be the w-canonical set for U. If $Z(U) \cap \{x_j\}_{j=1}^n$ consists of k points, then $\{x_j\}_{j=1}^n$ has rank $n-k$.*

In particular, the only way in which a w-canonical set for a WT-space with the splitting property can fail to have full rank is if it contains common zeros of the space U. Not surprisingly, sufficiency in Theorem 1.1c follows readily from Theorem 1.3. In Section 3, we prove these as well as Theorem 1.1a.

The literature on uniqueness of best approximations contains numerous articles on the barycentric dimension of sets of best approximations (see [4, 15], and references therein). The following corollary is an application of Theorem 1.3 and the relation between best $\|\cdot\|_w$-approximation and interpolation.

COROLLARY 1.4. *Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property, $w \in C^+$, and $\{x_j\}_{j=1}^n$ be the w-canonical set for U. If*

$Z(U) \cap \{x_j\}_{j=1}^n$ consists of k points, then for every $f \in C(U)$, the set of best $\|\cdot\|_w$-approximations to f from U has barycentric dimension k or less.

In Section 4, we establish necessity in Theorem 1.1b, c.

Much of the discussion above involved WT-extensions of a WT-space U (that is, nontrivial elements of $C(U)$). The concept of extending spaces with a given property has garnered considerable attention. Specifically, extension of n-dimensional Chebyshev (T-) spaces (that is, no nontrivial element can have more than $n-1$ zeros) to $(n+1)$-dimensional T-spaces was established by Zielke [18, 19] and Zalik [20]. In Section 5, we prove the extension result for A-spaces.

THEOREM 1.5. *Let U be an n-dimensional A-space in $C[a, b]$. Then there exists $f \in C[a, b]$ such that $U \oplus sp\{f\}$ is an $(n+1)$-dimensional A-space.*

2. INTERLACING RESULT

In this section, we prove sufficiency in Theorem 1.1b and Theorem 1.2 and demonstrate the sharpness of Theorem 1.2. In fact, we establish a lemma which yields both results in very simple fashions.

Before proceeding, we recall some terminology, notations, and two fundamental theorems on WT-spaces that are used throughout this paper. For r a positive integer, let $\Delta_r = \Delta_r(a, b) = \{(x_j)_{j=1}^r : a < x_1 < \cdots < x_r < b\}$. We say that a real-valued function u on $[a, b]$ *changes sign weakly* on $(x_j)_{j=1}^r \in \Delta_r$ if for $\gamma = 0$ or 1, $(-1)^{j+\gamma} u \geq 0$ on $[x_j, x_{j+1}]$ $(0 \leq j \leq r)$ where $x_0 = a$ and $x_{r+1} = b$. Also, if $u_1, ..., u_n$ are functions on $[a, b]$ and $x_1, ..., x_n \in [a, b]$, denote

$$D\begin{pmatrix} u_1, ..., u_n \\ x_1, ..., x_n \end{pmatrix} = \det[u_i(x_j)]_{i,j=1}^n.$$

The following result appears in Zielke [19, p. 12].

PROPOSITION 2.1. *An n-dimensional subspace U of $C[a, b]$ is a WT-space if and only if any one of the following equivalent conditions hold:*

1. *If $\{u_i\}_{i=1}^n$ is a basis for U, then for $\gamma = 1$ or -1,*

$$\gamma D\begin{pmatrix} u_1, ..., u_n \\ x_1, ..., x_n \end{pmatrix} \geq 0$$

for all $(x_j)_{j=1}^n \in \Delta_n$.

2. *If $(x_j)_{j=1}^{n-1} \in \Delta_{n-1}$, there exists $u \in U \backslash \{0\}$ that changes sign weakly on $(x_j)_{j=1}^{n-1}$.*

Actually, more characterizations of WT-spaces exist; we have stated only those that we shall use. The second theorem we cite is due to Stockenberg [17] and regards the number of certain zeros of functions in a WT-space. If U is a subspace of $C[a, b]$, we say that a point $x \in [a, b]$ is *essential* (with respect to U) if $x \notin Z(U)$. Also, if $u \in C[a, b]$, we say that a set $x_1 < \cdots < x_k$ of zeros of u are *separated* if $u \not\equiv 0$ on (x_i, x_{i+1}) $(1 \leqslant i \leqslant k-1)$.

PROPOSITION 2.2. *Let U be an n-dimensional WT-space in $C[a, b]$ and $u \in U$. If $x_1 < \cdots < x_n$ are essential, separated zeros of u, then $u \equiv 0$ on $[a, x_1]$ or on $[x_n, b]$. In particular, if $u \in U$ has n essential zeros in (a, b), then u has a zero interval.*

Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property. Fix $w \in C^+$, and let

$$a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$$

and $\sigma_1 = (-1)^j$ on $[x_j, x_{j+1})$ $(0 \leqslant j \leqslant n)$ where

$$\int_a^b \sigma_1 uw \, d\mu = 0 \tag{2.1}$$

for all $u \in U$. That is, $\{x_j\}_{j=1}^n$ is a w-canonical set *for* U. Now let

$$a = y_0 < y_1 < \cdots < y_m < y_{m+1} = b$$

and $\sigma_2 = (-1)^j$ on $[y_j, y_{j+1})$ $(0 \leqslant j \leqslant m)$, where

$$\int_a^b \sigma_2 uw \, d\mu = 0 \tag{2.2}$$

for all $u \in U$. Since U is a WT-space, $m \geqslant n$.

LEMMA 2.3. *Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property, and let $(x_j)_{j=1}^n$ and $(y_j)_{j=1}^m$ be as above. Then $y_j \leqslant x_j$ $(1 \leqslant j \leqslant n)$ and $y_{m+1-j} \geqslant x_{n+1-j}$ $(1 \leqslant j \leqslant n)$.*

Proof. The second set of inequalities follows from the first set by reversing the interval.

Suppose that the first set of inequalities is false. Let i be the first index in $\{1, ..., n\}$ where $y_i > x_i$. We consider two cases.

Case 1. Suppose that $y_k = x_k$ $(1 \leqslant k \leqslant i-1)$. By Proposition 2.1(2), we can find $u \in U \setminus \{0\}$ so that $\sigma_1 u \leqslant 0$ on $[a, x_i)$ and $\sigma_1 u \geqslant 0$ on $[x_i, b)$. Now $\sigma_1(\sigma_1 - \sigma_2) \geqslant 0$ and $\sigma_1 - \sigma_2 \equiv 0$ on $[a, x_i)$. So $(\sigma_1 - \sigma_2) u \geqslant 0$, and since

$$\int_a^b (\sigma_1 - \sigma_2) \, uw \, d\mu = 0,$$

$(\sigma_1-\sigma_2)\,u\equiv 0$. Since $y_i>x_i$ and $y_k=x_k$ $(0\leqslant k\leqslant i-1)$, $\sigma_1-\sigma_2$ is nonzero on $[x_i, x_i+\varepsilon)$ for some $\varepsilon>0$, and thus $u\equiv 0$ on $[x_i, x_i+\varepsilon]$. By the splitting property, $u\chi_{[a,x_i]}$, $u\chi_{[x_i,b]}\in U$. Since $\sigma_1(\sigma_1+\sigma_2)\geqslant 0$, $(\sigma_1+\sigma_2)\,u\chi_{[a,x_i]}\leqslant 0$ and $(\sigma_1+\sigma_2)\,u\chi_{[x_i,b]}\geqslant 0$. But by (2.1) and (2.2),

$$\int_a^b (\sigma_1+\sigma_2)\,u\chi_{[a,x_i]}\,w\,d\mu=\int_a^b (\sigma_1+\sigma_2)\,u\chi_{[x_i,b]}\,w\,d\mu=0,$$

and it follows that $(\sigma_1+\sigma_2)\,u\chi_{[a,x_i]}\equiv 0$ and $(\sigma_1+\sigma_2)\,u\chi_{[x_i,b]}\equiv 0$. So $(\sigma_1+\sigma_2)\,u\equiv 0$. It now follows that $u\equiv 0$, which is a contradiction.

Case 2. Suppose that $y_k<x_k$ for some $1\leqslant k\leqslant i-1$. Choose $j\in\{1,...,i-1\}$ so that $y_j<x_j$ and $y_k=x_k(j+1\leqslant k\leqslant i-1)$. Since U contains an $(n-1)$-dimensional WT-space (see [19, *p.* 31]), Proposition 2.1(2) yields $u\in U\backslash\{0\}$ such that $\sigma_1 u\geqslant 0$ on $[a, x_j)\cup[x_i, b)$ and $\sigma_1 u\leqslant 0$ on $[x_j, x_i)$. Since $\sigma_1-\sigma_2\equiv 0$ on $[x_j, x_i)$, $(\sigma_1-\sigma_2)\,u\geqslant 0$. As in Case 1, $(\sigma_1-\sigma_2)\,u\equiv 0$. But $\sigma_1-\sigma_2$ is nonzero on $(x_j-\varepsilon, x_j)\cup(x_i, x_i+\varepsilon)$ for some $\varepsilon>0$, and thus $u\equiv 0$ on this set. By the splitting property, $u\chi_{[a,x_j]}$, $u\chi_{[x_j,x_i]}$, $u\chi_{[x_i,b]}\in U$. As in Case 1, $(\sigma_1+\sigma_2)\,u\chi_{[a,x_j]}\equiv(\sigma_1+\sigma_2)\,u\chi_{[x_j,x_i]}\equiv(\sigma_1+\sigma_2)\,u\chi_{[x_i,b]}\equiv 0$ so that $(\sigma_1+\sigma_2)\,u\equiv 0$, and $u\equiv 0$, a contradiction. ∎

Lemma 2.3 provides a very simple proof of sufficiency in Theorem 1.1b.

Proof of sufficiency for Theorem 1.1b. We let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property and $w\in C^+$. Let $(x_j)_{j=1}^n$, $(y_j)_{j=1}^n\in\Delta_n$ be w-canonical sets for U. Particularly, (2.1) and (2.2) hold where $m=n$. By Lemma 2.3, $y_j\leqslant x_j$ $(1\leqslant j\leqslant n)$ and $y_{n+1-j}\geqslant x_{n+1-j}$ $(1\leqslant j\leqslant n)$. That is, $y_j=x_j$ $(1\leqslant j\leqslant n)$, and the w-canonical set for U is unique. ∎

In a similar fashion, the interlacing result evolves.

Proof of Theorem 1.2. We apply Lemma 2.3 where $m=n+1$. Then $y_j\leqslant x_j$ $(1\leqslant j\leqslant n+1)$ and $y_{n+2-j}\geqslant x_{n+1-j}$ $(1\leqslant j\leqslant n+1)$. (Note that $y_{n+1}<x_{n+1}=b$ and $y_1>x_0=a$.) The latter inequality becomes $y_i\geqslant x_{i-1}$ $(1\leqslant i\leqslant n+1)$ by letting $i=n+2-j$. ∎

We conclude this section with two examples. The first demonstrates that we cannot obtain strict interlacing in Theorem 1.2 even if U is an A-space, and the second shows that we cannot remove the splitting property in Theorem 1.2.

EXAMPLE 1. Let $a<c<b$ and U be an n-dimensional A-space in $C[a, b]$ where $u\equiv 0$ on $[c, b]$ for all $u\in U$. Evidently, the 1-canonical set $\{x_j\}_{j=1}^n$ for U is contained in the interval (a, c). Let $f\in C[a, b]$ where $f\equiv 0$ on $[a, c]$ and $f>0$ on (c, b). Then $V=U\oplus sp\{f\}$ is a WT-extension of U.

In fact, V is an A-space. The 1-canonical set for V is $\{x_j\}_{j=1}^{n+1}$ where $x_{n+1} \in (c, b)$ and $\int_c^{x_{n+1}} f\,d\mu = \int_{x_{n+1}}^b f\,d\mu$. Hence, strict interlacing fails.

EXAMPLE 2. Let $U = sp\{u_1\}$ where $u_1 \in C[0, 4]$, $u \equiv 0$ on $[1, 2] \cup [3, 4]$, $u > 0$ on $(0, 1) \cup (2, 3)$, and $\int_0^1 u_1\,d\mu = \int_2^3 u_1\,d\mu$. Clearly, U is a WT-space that does not satisfy the splitting property and the 1-canonical sets for U are of the form $\{x_1\}$ where $1 \leqslant x_1 \leqslant 2$. Let $V = sp\{u_1, u_2\}$ where $u_2 \equiv 0$ on $[0, 3]$ and $u_2 > 0$ on $(3, 4)$. Then V is a WT-extension of U, and the 1-canonical sets for V are of the form $\{x_1, x_2\}$ where $1 \leqslant x_1 \leqslant 2$ and x_2 is the unique point in $(3, 4)$ where $\int_3^{x_2} u_2\,d\mu = \int_{x_2}^4 u_2\,d\mu$. Now $\{1\}$ and $\{2, x_2\}$ are 1-canonical sets for U and V, respectively, that fail to interlace.

The authors do not know whether for an arbitrary WT-space there exists a w-canonical set that interlaces with the w-canonical sets for its WT-extensions ($w \in C^+$).

3. Rank of the w-Canonical Sets

In this section, we prove Theorem 1.3 after a short development during which the proof of Theorem 1.1a arises. In addition, sufficiency in Theorem 1.1c results.

We first give a lemma that identifies when a set of points is a w-canonical set for a space U and some weight function $w \in C^+$.

LEMMA 3.1. *Let U be an n-dimensional subspace of $C[a, b]$, and let $(x_j)_{j=1}^k \in \Delta_k$ where $0 \leqslant k \leqslant n$. Then $\{x_j\}_{j=1}^k$ is a w-canonical set for U for some $w \in C^+$ if and only if there does not exist $v \in U \setminus \{0\}$ that changes sign weakly on $(x_j)_{j=1}^k$.*

Proof. If such a $v \in U \setminus \{0\}$ does exist, then (1.1) clearly fails for $u = v$ and all $dv = w\,d\mu (w \in C^+)$.

To prove the other direction, we use a proposition on moments (see [4, 13]).

PROPOSITION 3.2. *Let V be a finite dimensional subspace of $L^\infty[a, b]$. If V does not contain a nontrivial element that is nonnegative μ-a.e., then there exists $w \in C^+$ so that $\int_a^b uw\,d\mu = 0$ for all $u \in V$.*

We suppose that there is no $v \in U \setminus \{0\}$ that changes sign weakly on $(x_j)_{j=1}^k$. Define $\sigma : [a, b] \to \{-1, 1\}$ by $\sigma = (-1)^i$ on $[x_i, x_{i+1})$ $(0 \leqslant i \leqslant k-1)$ and $\sigma = (-1)^k$ on $[x_k, x_{k+1}]$, where $x_0 = a$ and $x_{k+1} = b$. By hypothesis the space σU contains no nontrivial element that is nonnegative μ-a.e., and thus by Proposition 3.2 there exists $w \in C^+$ so that $\int_a^b \sigma uw\,d\mu = 0$ for all $u \in U$. That is, $\{x_j\}_{j=1}^k$ is a w-canonical set for U.

We remark here that Theorem 1.1a follows immediately from Lemma 3.1 and Proposition 2.1.

Our next lemma gives further insight into the w-canonical sets for WT-spaces.

LEMMA 3.3. *Let U be an n-dimensional WT-space in $C[a, b]$. Then $\{(x_j)_{j=1}^n \in \Delta_n : \{x_j\}_{j=1}^n$ is a w-canonical set for U for some $w \in C^+\}$ is an open subset of $\Delta_n \subseteq \mathbb{R}^n$.*

Proof. Assume this set is not open in Δ_n. Then there exist $\tilde{w} \in C^+$, a $\tilde{w}$-canonical set $(x_j)_{j=1}^n \in \Delta_n$ for U, and a sequence $(x_j^k)_{j=1}^n \in \Delta_n$ $(k = 1, 2, ...,)$ such that $x_j^k \to x_j$ $(1 \leqslant j \leqslant n)$ and each $(x_j^k)_{j=1}^n$ is not a w-canonical set for any $w \in C^+$. By Lemma 3.1, for each k there is a $v_k \in U \backslash \{0\}$ with $\|v_k\|_\infty = 1$ such that v_k changes sign weakly on $(x_j^k)_{j=1}^n$. ($\|\cdot\|_\infty$ denotes the uniform norm over $[a, b]$.) We may assume that $v_k \to v$ uniformly where $v \in U \backslash \{0\}$, and it follows that v changes sign weakly on $(x_j)_{j=1}^n$ contrary to Lemma 3.1. ∎

Our final lemma in the development of Theorem 1.3 is a precursor of this theorem.

LEMMA 3.4. *Let U be an n-dimensional WT-space in $C[a, b]$ with the splitting property and $w \in C^+$. If $(x_j)_{j=1}^n \in \Delta_n$ is the w-canonical set for U and has rank less than n, then $Z(U) \cap \{x_j\}_{j=1}^n \neq \emptyset$.*

To prove Lemma 3.4, we need a result of Li [8] regarding splitting of certain subspaces of WT-spaces.

PROPOSITION 3.5. *Let U be an n-dimensional WT-space in $C[a, b]$ and $a \leqslant c < z < d \leqslant b$. If $z \in Z(U_{[c,d]}) \backslash Z(U)$, then $U_{[c,d]} = U_{[c,z]} \oplus U_{[z,d]}$.*

Proof of Lemma 3.4. Assume that $(x_j)_{j=1}^n$ has rank less than n. Thus $\dim U|_{\{x_j\}_{j=1}^n} < n$, and thus there exists $v \in U \backslash \{0\}$ such that $v(x_j) = 0$ $(1 \leqslant j \leqslant n)$. Now assume that $Z(U) \cap \{x_j\}_{j=1}^n = \emptyset$. By Proposition 2.2, v necessarily vanishes on an interval. By the splitting property and since $\dim U < \infty$, we may assume that $v \in U_J$ where J is a closed subinterval of $[a, b]$ and v has no zero intervals in J. If some $x_j \in Z(U_J) \cap \operatorname{Int} J$, then by Proposition 3.5, U_J splits. Thus we have closed intervals $G_1 \leqslant \cdots \leqslant G_s$ contained in J where $v \in U_{G_1} \oplus \cdots \oplus U_{G_s}$, v has no zero intervals in G_i $(1 \leqslant i \leqslant s)$, and $Z(U_{G_i}) \cap \operatorname{Int} G_i \cap \{x_j\}_{j=1}^n = \emptyset$ $(1 \leqslant i \leqslant n)$.

Now each U_{G_i} is a WT-space (see [7, 8]), and $\#(\operatorname{Int} G_i \cap \{x_j\}_{j=1}^n) \geqslant \dim U_{G_i}$ for otherwise $\operatorname{Int} G_i \cap \{x_j\}_{j=1}^n$ would be a w-canonical set for U_{G_i} with fewer than $\dim U_{G_i}$ points contrary to Theorem 1.1a. But now $\operatorname{Int} G_i \cap \{x_j\}_{j=1}^n$ constitutes $\dim U_{G_i}$ or more essential, separated zeros of $v\chi_{G_i} \in U_{G_i}$, where $v\chi_{G_i}$ has no zero intervals in G_i. This violates Proposition 2.2, and a contradiction is reached. ▮

We are now prepared to prove Theorem 1.3.

Proof of Theorem 1.3. We are given $w \in C^+$ and a w-canonical set $\{x_j\}_{j=1}^n$ for U containing precisely k points in $Z(U)$. Without loss of generality, $x_1, ..., x_k \in Z(U)$ and $x_{k+1}, ..., x_n \notin Z(U)$ $(1 \leqslant k \leqslant n-1)$. The case for $k=0$ is contained in Lemma 3.4. Assume that $\dim U|_{\{x_j\}_{j=k+1}^n} = \dim U|_{\{x_j\}_{j=1}^n} < n-k$. Since U is a WT-space with the splitting property, the w-canonical set for U is unique. Thus each $x_i \notin \operatorname{Int} Z(U)$. Using this and Lemma 3.3, we can find x_i^* $(1 \leqslant i \leqslant k)$ and $\tilde{w} \in C^+$ so that $\{x_1^*, ..., x_k^*, x_{k+1}, ..., x_n\}$ is the $\tilde{w}$-canonical set for U and is disjoint with $Z(U)$. By Lemma 3.4, $\{x_1^*, ..., x_k^*, x_{k+1}, ..., x_n\}$ has rank n. But since $\dim U|_{\{x_j\}_{j=1}^n} < n-k$, $\{x_1^*, ..., x_k^*, x_{k+1}, ..., x_n\}$ has rank less than n, a contradiction. ▮

We now turn our attention to the proof of sufficiency of Theorem 1.1c. Sufficiency traces back to Sommer [16] in a considerably different fashion.

Proof of sufficiency for Theorem 1.1c. We assume that U is an A-space; that is, it satisfies the WT-, splitting, and decomposition properties. Let $w \in C^+$. By sufficiency in Theorem 1.1a, b, there is a unique w-canonical set $(x_j)_{j=1}^n$ for U. We show that $(x_j)_{j=1}^n$ has rank n. Assume it has rank less than n. By Lemma 3.4, some $x_i \in Z(U)$, and by the decomposition property, $U = U_{[a, x_i]} \oplus U_{[x_i, b]}$. Now $\{x_j\}_{j=1}^{i-1}$ is a w-canonical set for $U_{[a, x_i]}$ and $\{x_j\}_{j=i+1}^n$ is a w-canonical set for $U_{[x_i, b]}$. But $i-1 < \dim U_{[a, x_i]}$ or $n-i < \dim U_{[x_i, b]}$ which is a contradiction in view of Theorem 1.1a and the fact that $U_{[a, x_i]}$ and $U_{[x_i, b]}$ are WT-spaces [7, 8]. ▮

4. Necessity for Theorem 1.1b, c

In Theorem 1.1 it remains to prove necessity in (b) and (c), and we do so in this section. We use a result of Kroó [6] that characterized those WT-spaces that have locally unique w-canonical sets.

Proposition 4.1. *Let U be an n-dimensional WT-space in $C[a, b]$. The following are equivalent*:

(a) *for all $w \in C^+$, the w-canonical sets are locally unique*;

(b) *given $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, where $n-r+1$ of the points in $\{x_j\}_{j=1}^n$ are contained in common zero intervals of r linearly* *independent functions in* U $(1 \leqslant r \leqslant n)$, *there exists* $v \in U \backslash \{0\}$ *that changes sign weakly on* $\{x_j\}_{j=1}^n$.

Kroó [6] enquired whether condition (b) is sufficient for global uniqueness. In the next theorem, we show that this is indeed the case.

THEOREM 4.2. *Let* U *be an n-dimensional WT-space in* $C[a, b]$. *The following are equivalent*:

(a) *for all* $w \in C^+$, *the w-canonical sets are unique*;

(b) *for all* $w \in C^+$, *the w-canonical sets are locally unique*;

(c) *given* $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, *where* $n - r + 1$ *of the points in* $\{x_j\}_{j=1}^n$ *are contained in common zero intervals of* r *linearly independent functions in* $U (1 \leqslant r \leqslant n)$, *there exists* $v \in U \backslash \{0\}$ *that changes sign weakly on* $\{x_j\}_{j=1}^n$;

(d) U *satisfies the splitting property*.

Necessity in Theorem 1.1b follows readily from Theorems 1.1a and 4.2.

Proof of Theorem 4.2. Proposition 4.1 yields the equivalence of (b) and (c), while it is clear that (a) implies (b). Further, sufficiency in Theorem 1.1b provides that (d) implies (a). Thus, we need only prove that (c) implies (d). We assume that U is a *WT*-space and that condition (c) holds, and establish that U satisfies the splitting property.

Let $g \in U \backslash \{0\}$ where $g \equiv 0$ on $[c, d]$ $(a < c < d < b)$. Let $V = \{u \in U : u \equiv 0 \text{ on } [c, d]\}$ and let W complement V in U. Since $g \in V$, $\dim V = k \geqslant 1$ and $\dim W = \dim X|_{(c,d)} = n - k$. Choose a set $S_1 \subseteq (a, c) \cup (d, b)$ of k points and a set $S_2 \subseteq (c, d)$ of $n - k$ points where $\dim V|_{S_1} = k$ and $\dim W|_{S_2} = n - k$. Write $S_1 = \{y_j\}_{j=1}^{v} \cup \{y_j\}_{j=\mu+1}^{n}$ $(k = n - \mu + v)$ and $S_2 = \{y_j\}_{j=v+1}^{\mu}$ $(n - k = \mu - v)$, where $a < y_1 < \cdots < y_v < c < y_{v+1} < \cdots < y_\mu < d < y_{\mu+1} < \cdots < y_n < b$. Now choose bases $\{u_i\}_{i=1}^{v} \cup \{u_i\}_{i=\mu+1}^{n}$ for V and $\{u_i\}_{i=v+1}^{\mu}$ for W satisfying

$$u_i(y_j) = \delta_{ij} \qquad (i = 1, ..., v, \mu + 1, ..., n; j = 1, ..., n) \tag{4.1}$$

and

$$u_i(y_j) = \delta_{ij} \qquad (i, j = v + 1, ..., \mu), \tag{4.2}$$

where δ_{ij} denotes the Kronecker delta function.

It suffices to prove that $u_i \equiv 0$ on $[c, b]$ $(1 \leqslant i \leqslant v)$ and $u_i \equiv 0$ on $[a, d]$ $(\mu + 1 \leqslant i \leqslant n)$. We prove the first; the second is similar.

Fix $1 \leqslant i \leqslant v$. Observe that

$$D\begin{pmatrix} u_1, ..., u_n \\ y_1, ..., y_n \end{pmatrix} = 1$$

and by Proposition 2.1,

$$D\begin{pmatrix} u_1, \dots, u_n \\ t_1, \dots, t_n \end{pmatrix} \geqslant 0 \tag{4.3}$$

for all $(t_j)_{j=1}^n \in \Delta_n$. Furthermore,

$$u_i(x) = D\begin{pmatrix} u_1, \dots, u_i, \dots, u_n \\ y_1, \dots, x, \dots, y_n \end{pmatrix},$$

and by (4.3) u_i changes sign weakly on $\{y_j\}_{j=1, j\neq i}^n$.

Now c and y_j $(j=1, \dots, n; j \neq i)$ constitute n points in (a, b) of which $n-k+1$ of them $(c, y_{\nu+1}, \dots, y_\mu)$ lie in a common zero interval of k linearly independent functions in $U(u_1, \dots, u_\nu, u_{\mu+1}, \dots, u_n)$. By Proposition 4.1, there exists $v \in U \backslash \{0\}$ that changes sign weakly on the set $\{c\} \cup \{y_j\}_{j=1, j\neq i}^n$. But $v(y_j) = 0$ $(j=1, \dots, n, j \neq i)$, and since $\{u_j\}_{j=1}^n$ is a basis for U, (4.1) and (4.2) imply that $v = \alpha u_i$ for some $\alpha \neq 0$. Hence, u_i changes sign weakly on $\{c\} \cup \{y_j\}_{j=1, j\neq i}^n$.

Since u_i changes sign weakly on $\{y_j\}_{j=1, j\neq i}^n$ and on $\{c\} \cup \{y_j\}_{j=1, j\neq i}^n$, it now follows that $u_i \equiv 0$ on $[c, b]$. ∎

Before proving necessity for Theorem 1.1c, we establish a lemma that is of interest in itself. As with Li's result, Proposition 3.5, it gives a condition under which a WT-space decomposes.

LEMMA 4.3. *Let U be an n-dimensional WT-space in $C[a, b]$ and $z \in Z(U) \cap (a, b)$. If, for every $w \in C^+$, z is not contained in a w-canonical set for u, then $U = U_{[a,z]} \oplus U_{[z,b]}$.*

Proof. The proof is similar to that of necessity for Theorem 1.1b. Choose $a < y_1 < \cdots < y_r < z < y_{r+1} < \cdots < y_n < b$ so that $\dim U|_{\{y_j\}_{j=1}^n} = n$ and the basis $\{u_i\}_{i=1}^n$ for U satisfying $u_i(x_j) = \delta_{ij}$ $(i, j = 1, \dots, n)$. For fixed $1 \leqslant i \leqslant r$, the argument above shows that u_i changes sign weakly on $\{y_j\}_{j=1, j\neq i}^n$. However, by hypothesis, $\{z\} \cup \{y_j\}_{j=1, j\neq i}^n$ is not a w-canonical set for U for any $w \in C^+$, and by Lemma 3.1 some $v \in U \backslash \{0\}$ changes sign weakly on $\{z\} \cup \{y_j\}_{j=1, j\neq i}^n$. As above, u_i changes sign weakly on this set. Thus $u_i \equiv 0$ on $[z, b]$. Similarly, $u_i \equiv 0$ on $[a, z]$ $(r+1 \leqslant i \leqslant n)$, and Lemma 4.3 follows. ∎

Proof of necessity for Theorem 1.1c. We assume that for every $w \in C^+$, the w-canonical set for U contains n points, is unique, and has rank n. By Theorem 1.1a, b, U satisfies the WT- and splitting properties. To prove the decomposition property, let $z \in Z(U) \cap (a, b)$. Since every w-canonical set for U has rank n, z is not in any w-canonical set for $U(w \in C^+)$, and by Lemma 4.3, $U = U_{[a,z]} \oplus U_{[z,b]}$. Thus U satisfies the decomposition property. ∎

5. Proof of Theorem 1.5

In this section, we prove the extension result Theorem 1.5 for A-spaces. Actually, we prove an extension result for nondecomposing A-spaces from which Theorem 1.5 readily follows.

Theorem 5.1. *Let U be an n-dimensional A-space in $C[a, b]$ where $Z(U) \cap (a, b) = \varnothing$. Then there exists $f \in C[a, b]$ such that*

(a) $f(a) = 0$

(b) $f(b) = 0$ if $b \in Z(U)$

(c) $U \oplus sp\{f\}$ *is an $(n+1)$-dimensional A-space in $C[a, b]$.*

To see that Theorem 1.5 follows from Theorem 5.1, the decomposition property for an A-space U in $C[a, b]$ allows us to write $U = U_{J_1} \oplus \cdots \oplus U_{J_r}$, where $J_1, ..., J_r$ are closed intervals, $J_1 \leqslant \cdots \leqslant J_r$, and $Z(U_{J_i}) \cap \operatorname{Int} J_i = \varnothing$ $(1 \leqslant i \leqslant r)$. Each U_{J_i} is an A-space. Applying Theorem 5.1 to U_{J_i} yields $f \in C(J_i)$ which is an A-space extension of U_{J_i}. Parts (a) and (b) of Theorem 5.1 allow us to extend f continuously to $[a, b]$ by $f \equiv 0$ on $[a, b] \backslash J_i$ and the result is easily seen to be an A-space extension of U.

To prove Theorem 5.1, we first isolate an extension lemma.

Lemma 5.2. *Let U be an n-dimensional subspace of $C[a, b]$ $(n \geqslant 2)$ satisfying the T-property on (a, b), and let $a < c < b$. Then there exists $f \in C[a, b]$ such that*

(a) $f \equiv 0$ on $[a, c]$

(b) $f|_{[c,b]} \in U|_{[c,b]}$

(c) $U \oplus sp\{f\}$ *is an $(n+1)$-dimensional WT-space in $C[a, b]$.*

Proof. Consider the Gauss transformation $L_k : U \to C[a, b]$ by

$$(L_k u)(x) = \int_a^b u(s) \frac{k}{\sqrt{2\pi}} e^{-k^2(s-x)^2/2} \, ds.$$

From [2, p. 15], $L_k u \to u$ pointwise on (a, b) as $k \to \infty$ for all $u \in U$ and $U_k := \{L_k u : u \in U\}$ is an n-dimensional extended Chebyshev (ET-) space on (a, b). Let $\{u_i\}_{i=1}^n$ be a basis for U.

For fixed k, select $g_k \in U_k \backslash \{0\}$ so that $g_k(c) = g_k'(c) = \cdots = g_k^{(n-2)}(c) = 0$ and write

$$g_k = \sum_{i=1}^n \alpha_i^k L_k u_i,$$

where $\max_{1 \leqslant i \leqslant n} |\alpha_i^k| = 1$. Let $f_k = g_k \chi_{[c,b]}$. By Theorem 17.1, p. 72, in Zielke [19], $U_k \oplus sp\{f_k\}$ is an $(n+1)$-dimensional WT-space in $C[a, b]$. (To fit Zielke's condition, we would first need to extract a basis for U which is a Markoff system on (a, b). This is possible.) Now extract a subsequence and relabel so that $\alpha_i^k \to \alpha_i$ $(1 \leqslant i \leqslant n)$ as $k \to \infty$ where $\max_{1 \leqslant i \leqslant n} |\alpha_i| = 1$. Let

$$g = \sum_{i=1}^{n} \alpha_i u_i$$

and $f = g\chi_{[c,b]}$.

Since $g_k \to g$ pointwise on (a, b) and $n \geqslant 2$, $g(c) = 0$ so that $f \in C[a, b]$. Now (a) and (b) are clear. For (c), $\max_{1 \leqslant i \leqslant n} |\alpha_i| = 1$ implies that $g \not\equiv 0$, and since U is a T-space in (a, b), $f \not\equiv 0$. Again since U is a T-space on (a, b), $U + sp\{f\}$ has dimension $n+1$, and that $U \oplus sp\{f\}$ is a WT-space follows from the fact that $U_k \oplus sp\{f_k\}$ is a WT-space, the pointwise convergences $L_k u_i \to u_i$ and $f_k \to f$ on (a, b), and Proposition 2.1. ∎

Proof of Theorem 5.1. Let U be an n-dimensional A-space in $C[a, b]$ where $Z(U) \cap (a, b) = \varnothing$. Let $a = d_0 < d_1 < \cdots < d_l < d_{l+1} = b$ be the endpoints of zero intervals of elements of U. Pinkus [11, 12] has proven that there are only finitely many such points and that $U|_{(d_i, d_{i+1})}$ is a T-space $(0 \leqslant i \leqslant l)$. We consider two cases.

Case 1. Suppose $\dim U|_{(d_l, b)} = 1$. Choose $g \in U$ so that $g \not\equiv 0$ on (d_l, b). Define $f \in C[a, b]$ by

$$f(x) = (x - d_l)\, g(x)\, \chi_{[d_l, b]}(x).$$

Evidently, $f \in C[a, b]$, $f(a) = 0$, and $f(b) = 0$ if $g(b) = 0$. Let $\tilde{U} = U \oplus sp\{f\}$. Since $U|_{(d_l, b)}$ is a 1-dimensional T-space, g does not vanish on (d_l, b), and thus $\tilde{U}|_{(d_l, b)} = g|_{(d_l, b)}\, sp\{1, x - d_l\}$ is a 2-dimensional T-space. Thus $\dim \tilde{U} = n+1$. If $l = 0$, $\tilde{U}$ is a T-space on (a, b) and therefore is an A-space. Suppose $l \geqslant 1$. Clearly, $Z(\tilde{U}) \cap (a, b) = \varnothing$ and $\tilde{U}$ satisfies the splitting property. To show that $\tilde{U}$ is a WT-space, suppose that $\tilde{u} \in \tilde{U} \setminus \{0\}$ has a strong alternation of length $n+2$. Since $g(d_l) \neq 0$, we can replace $\tilde{u}$ by $\tilde{u} + \varepsilon g$ for $\varepsilon \neq 0$ and sufficiently small and thus assume that $\tilde{u}(d_l) \neq 0$. Thus we may assume that d_l is one of the $n+2$ points of strong alternation of $\tilde{u}$. Thus the number of points of this strong alternation in $[a, d_l]$ is at least $n+1$ or the number of points of this strong alternation in $[d_l, b]$ is at least 3. Either way, we obtain a contradiction since U is an n-dimensional WT-space and $\tilde{U}|_{[d_l, b]}$ is a 2-dimensional WT-space. Thus $\tilde{U}$ is a WT-space, and by the Pinkus–Li result [8, 11, 12], $\tilde{U}$ is an A-space.

Case 2. We suppose that $\dim U|_{[d_l, b]} \geqslant 2$. Let $c_i = d_i$ $(0 \leqslant i \leqslant l)$, $d_l < c_{l+1} < b$, and $c_{l+2} = b$. By Lemma 5.2, select $f \in C[a, b]$ so that

(a) $f \equiv 0$ on $[a, c_{l+1}]$

(b) $f|_{[c_{l+1}, b]} \in U|_{[c_{l+1}, b]}$

(c) $U|_{[c_l, b]} \oplus sp\{f|_{[c_l, b]}\}$ is a WT-space of dimension $1 + \dim U|_{[c_l, b]}$.

We induct on l to prove that $\hat{U} := U \oplus sp\{f\}$ is an $(n+1)$-dimensional A-space in $C[a, b]$. By (c), $\dim \hat{U} = n+1$. For $\hat{U}$, the decomposition properly holds vacuously and the splitting property is obvious. It remains to show that $\hat{U}$ satisfies the WT-property. For $l = 0$, (c) implies that $\hat{U}$ is a WT-space and thus is an A-space.

Let $\eta = \dim U|_{[c_1, b]}$. Since $U|_{[c_1, b]}$ is an A-space of dimension η, the induction hypothesis implies that $\hat{U}|_{[c_1, b]}$ is an $(\eta+1)$-dimensional A-space. To prove that $\hat{U}$ is a WT-space, we use Proposition 2.1 and thus show that if $(x_j)_{j=1}^n \in \Delta_n$, then there is a $\hat{v} \in \hat{U} \setminus \{0\}$ that changes sign weakly on $X = \{x_j\}_{j=1}^n$. If $\dim U|_X < n$, then some $u \in U \setminus \{0\}$ vanishes on X and the definition of the A-property yields a $v \in U \setminus \{0\} \subseteq \hat{U} \setminus \{0\}$ that changes sign weakly on X. Thus we consider only the case where $\dim U|_X = n$.

Let $\Delta_n' = \{X \in \Delta_n : \dim U|_X = n\}$. Evidently, Δ_n' is open in $\mathbb{R}^n$. Hereafter, we use the symbol X for the n-tuple $(x_j)_{j=1}^n$ and its range $\{x_j\}_{j=1}^n$. No confusion will result.

For $X = (x_j)_{j=1}^n \in \Delta_n'$, let v_X denote the unique function in U satisfying $v_X(x_j) = f(x_j)$ $(1 \leqslant j \leqslant n)$, and let $\hat{v}_X = f - v_X$. For $1 \leqslant i \leqslant n$, let u_i be the unique element of U satisfying $u_i(x_j) = \delta_{ij}$ $(1 \leqslant j \leqslant n)$. Then $\{u_i\}_{i=1}^n$ is a basis for U,

$$u_i(x) = D\begin{pmatrix} u_1, \ldots, u_i, \ldots, u_n \\ x_1, \ldots, x, \ldots, x_n \end{pmatrix} \tag{5.1}$$

for $1 \leqslant i \leqslant n$, and

$$v_X(x) = \sum_{i \in K} f(x_i)\, u_i(x), \tag{5.2}$$

where $K = \{j : x_j \in (c_{l+1}, b)\}$. Since U is a WT-space, (5.1) implies that each u_i changes sign weakly on $X \setminus \{x_i\}$, and the uniqueness of u_i and v_X (as interpolants) and the splitting property for U imply that the closed support of each u_i and v_X is an interval.

We now break the proof into a sequence of lemmas.

LEMMA 5.3. *If $X \in \Delta_n'$ and $\#(X \cap (c_1, b)) = \eta$, then $\hat{v}_X|_{[c_1, b]}$ changes sign weakly on $X \cap (c_1, b)$.*

Proof. Write $X = (x_j)_{j=1}^n$ where $a < x_1 < \cdots < x_{n-\eta} \leqslant c_1 < x_{n-\eta+1} < \cdots < x_n < b$. Then $\{u_i|_{[c_1, b]}\}_{i=n-\eta+1}^n$ is a basis for $U|_{[c_1, b]}$ and for $x \in [c_1, b]$,

$$\hat{v}_X(x) = D\begin{pmatrix} u_{n-\eta+1}, \ldots, u_n, f \\ x_{n-\eta+1}, \ldots, x_n, x \end{pmatrix}.$$

Since $\hat{U}|_{[c_1,b]}$ is a WT-space, $\hat{v}_X|_{[c_1,b]}$ changes sign weakly on $X \cap (c_1, b)$. ∎

LEMMA 5.4. *Let* $X \in \Delta_n'$ *and* $a \leqslant c < d \leqslant b$. *Then* $\#(X \cap (c, d)) \geqslant \dim U_{[c,d]}$.

Proof. If $\#(X \cap (c, d)) < \dim U_{[c,d]}$, then some $u \in U_{[c,d]} \setminus \{0\}$ vanishes on $X \cap (c, d)$. By definition of $U_{[c,d]}$, u vanishes on X. This contradicts the fact that $X \in \Delta_n'$. ∎

The next lemma provides a means of creating new elements of Δ_n' from old ones.

LEMMA 5.5. *Let* $X = (x_j)_{j=1}^n \in \Delta_n'$, $1 \leqslant k \leqslant n$, *and* $Y = (X \setminus \{x_k\}) \cup \{\xi\}$, *where* $\xi \in \operatorname{Int} \overline{\operatorname{supp}}\, u_k \setminus X$. *Then* $Y \in \Delta_n'$.

Proof. Write $\overline{\operatorname{supp}}\, u_k = [c_p, c_q]$. Then $\xi \in (c_p, c_q)$, $u_k \in U_{[c_p,c_q]}$, and u_k has no zero intervals in $[c_p, c_q]$. Assume $Y \notin \Delta_n'$. Then some $v \in U \setminus \{0\}$ vanishes on Y. But by the nature of the basis $\{u_i\}_{i=1}^n$, $v = \alpha u_k$ for some nonzero constant α. Thus, u_k vanishes on Y. By Lemma 5.4, u_k has at least $\dim U_{[c_p,c_q]}$ separated zeros in (c_p, c_q). Now $U_{[c_p,c_q]}$ is an A-space (see [7]). Also, $Z(U_{[c_p,c_q]}) \cap (c_p, c_q) = \varnothing$, for otherwise, $U_{[c_p,c_q]}$ would decompose contradicting the uniqueness of u_k. Now the zero count for u_k as an element of $U_{[c_p,c_q]}$ violates Proposition 2.2. ∎

Of course, we prove that for each $X \in \Delta_n'$, $\hat{v}_X$ changes sign weakly on X. To this end, let $\Delta_n'' = \{X \in \Delta_n' : \hat{v}_X$ does not change sign weakly on $X\}$ and assume that $\Delta_n'' \neq \varnothing$. Since $\hat{v}_X$ depends continuously on X over Δ_n', the proof of Lemma 3.3 can be used to show that Δ_n'' is open. Let $m = \min\{\#(X \cap (c_{l+1}, b)) : X \in \Delta_n''\}$. Now $m > 0$ for if $X \cap (c_{l+1}, b) = \varnothing$, then $\hat{v}_X = f$ would change sign weakly on X. Let $\Delta_n''' = \{X \in \Delta_n'' : \#(X \cap (c_{l+1}, b)) = m$ and $c_{l+1} \notin X\}$. Since Δ_n'' is nonempty and open, Δ_n''' is also nonempty and open.

LEMMA 5.6. *If* $U \in \Delta_n'''$, *then* $\overline{\operatorname{supp}}\, \hat{v}_X$ *is* $[a, b]$ *or* $[c_1, b]$.

Proof. We first assert that $\hat{v}_X \not\equiv 0$ on $[c_{l+1}, b]$. Otherwise, we form Y by replacing one element of $X \cap (c_{l+1}, b)$ by c_{l+1}. By Lemma 5.5, $Y \in \Delta_n'$, and further $\hat{v}_Y = \hat{v}_X$ does not change sign weakly on Y. So $Y \in \Delta_n''$ which contradicts the minimality of m.

Next we show $\overline{\operatorname{supp}}\, \hat{v}_X = [c_i, b]$ for some $0 \leqslant i \leqslant l+1$. If this fails to hold, then since $\hat{v}_X \not\equiv 0$, $\hat{v}_X$ has a zero interval in $[c_j, c_{l+1}]$ for some $1 \leqslant j \leqslant l$ and is not identically zero on $[a, c_j]$. But $v_X \equiv -\hat{v}_X$ on $[a, c_{l+1}]$, and since $\overline{\operatorname{supp}}\, v_X$ is an interval, $v_X \equiv 0$ on $[c_j, b]$. But now 0 interpolates f on X contradicting the uniqueness of v_X.

Finally, we show that $i=0$ or 1. Suppose that $2 \leqslant i \leqslant l+1$. By the induction hypothesis, $\hat{U}|_{[c_1,b]}$ is an A-space and thus $(\hat{U}|_{[c_1,b]})_{[c_i,b]}$ is an A-space (see [7]). By the splitting property for $\hat{U}$, $\hat{U}_{[c_i,b]}$ is an A-space. Furthermore, $Z(\hat{U}_{[c_i,b]} \cap (c_i, b) = \varnothing$, for otherwise, $\hat{U}_{[c_i,b]}$ would decompose, violating the uniqueness of v_X. By Proposition 2.2, $\#(X \cap (c_i, b)) \leqslant \dim \hat{U}_{[c_i,b]} - 1$, and since $\hat{U}_{[c_i,b]}$ is a WT-space, there exists $\hat{w} \in \hat{U}_{[c_i,b]} \backslash \{0\}$ that changes sign weakly on $X \cap (c_i, b)$. Since $\hat{w} \equiv 0$ on $[a, c_i]$, $\hat{w}$ changes sign weakly on X. Now since $X \in \Delta_n'$, $\hat{w} = \alpha \hat{v}_X$ for some nonzero constant α. Hence, $\hat{v}_X$ changes sign weakly on X, which contradicts the fact that $X \in \Delta_n''$. ∎

LEMMA 5.7. *Let* $X = (x_j)_{j=1}^n \in \Delta_n'''$. *For each* $\varepsilon > 0$ *there exists* $Y = (y_j)_{j=1}^n \in \Delta_n'''$ *such that* $\max_{1 \leqslant j \leqslant n} |x_j - y_j| < \varepsilon$ *and* $\hat{v}_Y$ *has a sign change at each point in* $Y \cap \operatorname{Int} \overline{\operatorname{supp}}\, \hat{v}_Y$.

Proof. We first examine the case where $\hat{v}_X \not\equiv 0$ on $[a, c_1]$ so that by Lemma 5.6, $\hat{v}_X$ has no zero intervals. If $\hat{v}_X$ does not change sign at x_j, choose $\delta > 0$ so that $\hat{v}_X$ has constant nonzero sign on $(x_j - \delta, x_j + \delta) \backslash \{x_j\}$ and let σ_j be this sign. Since $X \in \Delta_n'$, we can find unique $w \in U$ so that for $1 \leqslant j \leqslant n$,

$$w(x_j) = \begin{cases} 0 & \text{if } \hat{v}_X \text{ changes sign at } x_j \\ -\sigma_j & \text{if } \hat{v}_X \text{ does not change sign at } x_j. \end{cases} \tag{5.3}$$

Take $\varepsilon > 0$ sufficiently small so that $x_j + \varepsilon < x_{j+1} - \varepsilon$ $(0 \leqslant j \leqslant n)$ where $x_0 = a$ and $x_{n+1} = b$, $x_{n-m} + \varepsilon < c_{l+1} < x_{n-m+1} - \varepsilon$, and if $y_j \in (x_j - \varepsilon, x_j + \varepsilon)$ $(1 \leqslant j \leqslant n)$, then $Y = (y_i)_{j=1}^n \in \Delta_n'''$. For $r > 0$ sufficiently small, $\hat{w} = \hat{v}_X + rw$ has a point of sign change in $(x_j - \varepsilon, x_j + \varepsilon)$ $(1 \leqslant j \leqslant n)$. For $1 \leqslant j \leqslant n$, let y_j be a point of sign change of $\hat{w}$ in $(x_j - \varepsilon, x_j + \varepsilon)$. Then $Y = (y_j)_{j=1}^n \in \Delta_n'''$ and $\hat{v}_Y = \hat{w}$ has a sign change at each y_j.

If $\hat{v}_X \equiv 0$ on $[a, c_1]$, choose w to satisfy (5.3) for $x_j \in (c_1, b)$ and $w(x_j) = 0$ for $x_j \in X \cap (a, c_1]$. Taking $y_j = x_j$ if $x_j \in (a, c_1]$ and the remaining y_j's as in the first case we obtain $Y \in \Delta_n'''$ with $\max_{1 \leqslant j \leqslant n} |x_j - y_j| < \varepsilon$ where $\hat{v}_Y$ changes sign at each $y_j \in (c_1, b)$. If $\hat{v}_Y \equiv 0$ on $[a, c_1]$, we are done. If $\hat{v}_Y \not\equiv 0$ on $[a, c_1]$, we simply apply the first case with X replaced with Y. ∎

LEMMA 5.8. *Let* $X \in \Delta_n'''$ *where* $\hat{v}_X$ *has a sign change at each point in* $X \cap \operatorname{Int} \overline{\operatorname{supp}}\, \hat{v}_X$. *Then* $\hat{v}_X|_{[a,c_{l+1}]}$ *changes sign weakly on* $X \cap (a, c_{l+1})$.

Proof. Since $\hat{v}_X$ has a sign change at each x_j in $X \cap \operatorname{Int} \overline{\operatorname{supp}}\, \hat{v}_X$, it suffices to show that $\hat{v}_X$ has no additional zeros in $(a, c_{l+1}) \cap \operatorname{Int} \overline{\operatorname{supp}}\, \hat{v}_X$. Suppose that $\hat{v}_X(\xi) = 0$ for some $\xi \in ((a, c_{l+1}) \cap \operatorname{Int} \overline{\operatorname{supp}}\, \hat{v}_X) \backslash X$. By (5.2), $\xi \in \operatorname{Int} \overline{\operatorname{supp}}\, u_k$ for some $x_k \in (c_{l+1}, b)$. By Lemma 5.5, $Y = (X \backslash \{x_k\}) \cup \{\xi\} \in \Delta_n'$. Furthermore $\hat{v}_Y = \hat{v}_X$ does not change sign weakly on Y since it changes sign at x_k. So $Y \in \Delta_n''$, which contradicts the minimality of m. ∎

Let $\mu = \max\{\#(X \cap (c_1, b)) : X \in \Delta_n'''\}$. By Lemma 5.4, $\#(X \cap (a, c_1)) \geqslant \dim U_{[a, c_1]} = n - \eta$ and thus $\#(X \cap (c_1, b)) \leqslant \eta$ for $X \in \Delta_n'$. Thus $\mu \leqslant \eta$.

LEMMA 5.9. $\mu = \eta$.

Proof. Assume that $\mu < \eta$. Choose $X \in \Delta_n'''$ where $\#(X \cap (c_1, b)) = \mu$. Since Δ_n''' is open, we may assume that $c_1 \notin X$. By Lemmas 5.7 and 5.8, we may assume that $\hat{v}_X|_{[a, c_{l+1}]}$ changes sign weakly on $X \cap (a, c_{l+1})$. Now $\#(X \cap (a, c_1)) = n - \mu > n - \eta = \dim U_{[a, c_1]}$. Thus for some $x_k \in (a, c_1)$, $u_k \not\equiv 0$ on $[c_1, b]$. By Lemma 5.6, we may choose $\xi \in (c_1, c_{l+1})$, where $\hat{v}_X(\xi)\, u_k(\xi) \neq 0$. By Lemma 5.5, $Y = (X \backslash \{x_k\}) \cup \{\xi\} \in \Delta_n'$. Now define $\sigma = (-1)^{j+v}$ on $[x_j, x_{j+1})$ $(0 \leqslant j \leqslant n)$ where $x_0 = a$ and $x_{n+1} = b$, and $v = \pm 1$ is chosen so that $\sigma(\xi)\, \hat{v}_X(\xi) > 0$. Recall that u_k changes weakly sign on $X \backslash \{x_k\}$. Let $\bar{u}_k = \pm u_k$ so that $\sigma \bar{u}_k \geqslant 0$ on $[x_k, b)$. Since $X \in \Delta_n'''$, $\sigma(x)\, \hat{v}_X(x) < 0$ for some $x \in (c_{l+1}, b)$.

Let $\hat{w} = \hat{v}_X - r\bar{u}_k$, where $r > 0$ is chosen so that $\hat{w}(\xi) = 0$. Then $\hat{w} = \hat{v}_Y$. Suppose that $\hat{w}$ changes sign weakly on Y. Note that $\sigma(x)\, \hat{w}(x) = \sigma(x)\, \hat{v}_X(x) - r\sigma(x)\, \bar{u}_k(x) < 0$ for x as above. Thus $\sigma\hat{\omega} \leqslant 0$ on $[\xi, b)$, $\sigma\hat{\omega} \geqslant 0$ on $[x_k, \xi)$, and $\sigma\hat{\omega} \leqslant 0$ on (a, x_k). But $\sigma(x_k)\, \hat{w}(x_k) = -r\sigma(x_k)\, \bar{u}_k(x_k) < 0$, a contradiction. Thus $\hat{w}$ does not change sign weakly on Y and so $y \in \Delta_n''$. But since $\xi \neq c_{l+1}$, $Y \in \Delta_n'''$ which contradicts the maximality of μ. ∎

To conclude the proof of Theorem 5.1, we employ Lemma 5.9 to obtain $X \in \Delta_n'''$ so that $\#(X \cap (c_1, b) = \eta$. By Lemmas 5.7 and 5.8, we may assume that $\hat{v}_X|_{[a, c_{l+1}]}$ changes sign weakly on $X \cap (a, c_{l+1})$. But by Lemma 5.3, $\hat{v}_X|_{[c_1, b]}$ changes sign weakly on $X \cap (c_1, b)$. Hence $\hat{v}_X$ changes sign weakly on X. This is a contradiction, and thus $\Delta_n'' = \varnothing$. Thus $\hat{U}$ is a WT-space. The proof is now complete. ∎

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